Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam January 2018: Problem 4 Solution

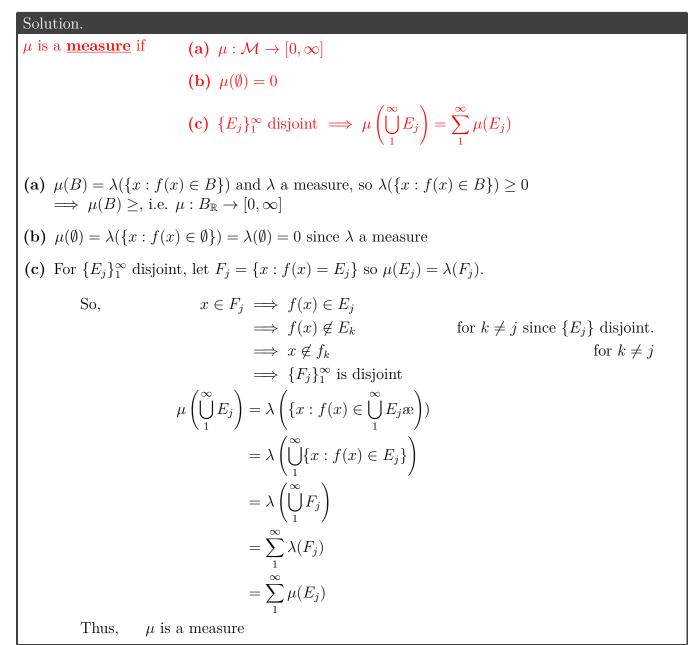
Exercise. Let λ denote Lebesuge measure on \mathbb{R} . Let $x \in \mathbb{R}$, and let $f : x \mapsto f(x) \in \mathbb{R}$ be Lebesgue measurable. For Borel sets $B \subset \mathbb{R}$, define

$$\mu(B) = \lambda(\{x : f(x) \in B\}).$$

Show that μ is a measure, and that

$$\int_{\mathbb{R}} g(y) d\mu(y) = \int_{\mathbb{R}} (g \circ f)(x) d\lambda(x)$$

for all g such that the integrals make sense.



Solution.

Show that

$$\int_{\mathbb{R}} g(y) d\mu(y) = \int_{\mathbb{R}} (g \circ f)(x) d\lambda(x)$$

for all g such that the integrals make sense.

Look at simple functions.

Let $\{\phi_n\}$ be a sequence of simple functions converging pointwise almost everywhere monotonically up to g, so by the Monotone Convergence Theorem,

$$\begin{split} \lim_{n \to \infty} \int \phi_n d\lambda &= \int g d\lambda \\ \phi_n &= \sum_{j=1}^{k_n} \alpha_{j,n} \chi(E_{j,n}) \\ \Longrightarrow & \int \int \phi d\lambda = \sum_{j=1}^{k_n} \alpha_{j,n} \lambda(E_{j,n}) \\ &= \sum_{j=1}^{k_n} \alpha_{j,n} \int E_{j,n} f d\mu \\ \int f \left(\sum_{j=1_n^k} \alpha_{j,n} \chi(E_{j,n}) \right) d\mu &= \int f \circ \phi_n d\mu \qquad f \circ \phi_n \text{ converges pointwise to } f \circ g \\ By \text{ MCT} & \int g d\lambda = \lim_{n \to \infty} \int f \circ \phi_n d\mu \\ &= \int f \circ g d\mu \end{split}$$